Essentials of Tropical Combinatorics by Michael Joswig Chapter 2.3 2.4

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Homogeneous Polynomials

Let $f \in \mathbb{K}[x_1^{\pm}, x_2^{\pm}, \ldots, x_d^{\pm}]$ be a homogeneous nonzero Laurent polynomial of degree δ. Then, for all $p \in \mathbb{R}^d$ and all $\lambda \in \mathbb{K}$,

$$f(\lambda p) = \lambda^{\delta} f(p).$$

Affine Hypersurface

The affine hypersurface of f, $V(f) \subset \mathbb{K}^d$, is a union of one-dimensional linear subspaces.

Projective Space

Definition

The set $\mathrm{PG}(\mathbb{K}^d)$ of linear subspaces of $\mathbb{K}^d$, equipped with the partial ordering by inclusion, is called the $(d-1)$-dimensional projective space over $\mathbb{K}$.

Projective Hypersurface

Definition

$V(f)$ is called the projective hypersurface in $\mathrm{PG}(\mathbb{K}^d)$, often written as $\mathrm{PG}_{d-1}\mathbb{K}$.

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This is because f is homogeneous. We apply the mapping from 1.4 into $d-1$ dimensions.

$$(x_1, x_2, \ldots, x_d) + \mathbb{R}\mathbb{1} \mapsto (x_2 - x_1, x_3 - x_1, \ldots, x_d - x_1)$$

Homogenous Version of Fundamental Theorem

Corollary

For a homogenous polynomial $f \in \mathbb{K}[x_1^{\pm}, x_2^{\pm}, \ldots, x_d^{\pm}]$ *the tropical hypersurface* $T(\mathrm{trop}(f))$ *equals the topological closure of the set* $\mathrm{ord}(V(f))$ *in* $\mathbb{R}^d / \mathbb{R}\mathbb{1}$.

Addition of Tropical Polynomials

Definition

The addition of tropical polynomials

$$F(X) = \oplus_{u \in S} a_u \odot X_1^{u_1} ... X_d^{u_d}, G(X) = \oplus_{v \in T} b_v \odot X_1^{v_1} ... X_d^{v_d}$$

with coefficients $a_b, b_b \in \mathbb{T}$ is defined as

$$(F \oplus G)(X) := \oplus_{w \in S \cup T} (a_w \oplus b_w) \odot X_1^{w_1} ... X_d^{w_d}$$

, where we take $a_u = \infty$ for $u \in T \setminus S$ and $b_v = \infty$ for $v \in S \setminus T$

Product of Tropical Polynomials

Definition

We then take the product of tropical polynomials to be

$$(F \odot G)(X) := \oplus_{u \in S} \oplus_{v \in T} (a_u \odot b_v) \odot X_1^{u_1+v_1} \ldots X_d^{u_d+v_d}$$

The Optimization Problem

We now explore the connection between tropical geometry and optimization. Ie. for $A = (a_{ij}) \in \mathbb{Z}^{d \times n}, b \in \mathbb{Z}^d, c \in \mathbb{R}^n$ we observe the optimization problem of minimizing $\langle c, x \rangle$. subject to $x \in \mathbb{N}^n, Ax = b$, an integer linear program.

Lemma 2.17

Assumption 2.16

We assume that the matrices A and the vector b are nonnegative, and assume $\exists\ \delta, k \in \mathbb{N}$ such that each column of A sums to δ, and the sum of the coefficients of b equals $k \cdot \delta$. z

Lemma 2.17

It turns out that any solution $x \in \mathbb{N}^n$ to the optimization problem satisfies $x_1 + x_2 ... + x_n = k$.

Theorem 2.18

Let F(X) denote the d-variate tropical polynomial $\bigoplus_{j=1}^{n} c_j \odot X_1^{a_{1j}} ... X_d^{a_{dj}}$. We note that this polynomial is homogeneous of degree δ (by the previous assumption)

Theorem

The optimal value of the integer linear program is the coefficient of the monomial $X_1^{b_1} \cdots X_d^{b_d}$ of $F(X)^{\odot k}$, which is kth power of the tropical polynomial F(X).

Example 2.19

Let

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix} \text{ and } b = \begin{pmatrix} 3 \\ 3 \end{pmatrix},$$

which yields $d = 2, n = d, \delta = 2$, and $k = 3$. Let $c = (0, 0, 1)$. To find the optimum integer solution $\mathbf{x}$, we construct the homogenous tropical polynomial below.

$$F(X, Y) = X^2 \oplus Y^2 \oplus 1 \otimes XY$$

Example 2.19

We wish to find the $k = 3$ power of F for our optimal solution (by theorem 2.18). Thus,

$$F(X, Y)^{\oplus 3} = X^6 \oplus X^5 Y \oplus X^4 Y^2 \oplus 1 \otimes X^3 Y^3 \oplus X^2 Y^4 \oplus 1 \otimes XY^5 \oplus Y^6.$$

The monomial $X^3 Y^3$ has a coefficient of 1. Thus,

$$1 = \min\{\langle c, x\rangle \mid x \in \mathbb{N}^n, Ax = b\},$$

and so $\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Problem 2.41

Let f and g be polynomials in $\mathbb{C}\{\{t\}\}[x_1^{\pm}, x_2^{\pm}, ..., x_d^{\pm}]$ such that $g \in \langle f \rangle$. Show that their tropical hypersurfaces satisfy $\mathcal{F}(f) \subseteq \mathcal{F}(g)$. When do we have equality?

Solution to Problem 2.41

For $f \in \mathbb{C}\{\{t\}\}[x_1^{\pm}, x_2^{\pm}, ..., x_d^{\pm}]$,

$$\langle f \rangle = \mathbb{K}[x_1^{\pm}, x_2^{\pm}, ..., x_d^{\pm}] \cdot f.$$

Thus, $V(g)$ contains the vanishing points of f. We have equality when $g = f$.

Problem 2.42

Prove the distributive law in the polytope algebra $\mathcal{B}_d$: For polynomials $P, Q, R \subset \mathbb{R}^d$ show that $(P \oplus Q) \odot R = (P \odot R) \oplus (Q \odot R)$ holds.