Chapter 7: Triangulations of Point Configurations

Sarah Pritchard and Nathan Stefanik

February 2021

Table of Contents

Point Configurations

Definition

(De Loera, et al.) A point configuration $\mathcal{V} = \{v_1, \ldots, v_n\} \subset \mathbb{R}^d$ is a finite set of (not necessarily distinct) points in Euclidean space $\mathbb{R}^d$.

Point Configurations

For convenience, we will consider **graded** point configurations in this chapter where all the points lie on the hyperplane $x_1 = 1$. We can convert any point configuration $\mathcal{V}$ into a graded point configuration by considering $\mathcal{A} = \{\binom{1}{v_1}, \binom{1}{v_2}, \cdots, \binom{1}{v_n}\}$.

Faces of Point Configurations

Let $\mathbf{a}_i$ denote the i-th element of $\mathcal{A}$ and $\sigma \subseteq [n]$ be a subset of indices. A subset $\mathcal{A}_\sigma$ of a point configuration is called a **face** of $\mathcal{A}$ if the elements of $\mathcal{A}_\sigma$ are precisely those that lie on a face of the polytope conv$(\mathcal{A})$.

Oriented Matroid

Definition

Two point configurations are **combinatorically equivalent** if they have identical sets of circuits up to relabeling.

Oriented Matroid

Definition

Two point configurations are **combinatorically equivalent** if they have identical sets of circuits up to relabeling.

This defines an equivalence relation on the set of all configurations, and the equivalence class of a configuration with respect to this equivalence relation is called its **oriented matriod**.

Subdivisions

A **subdivision** $\Delta = \{\sigma_1, \ldots, \sigma_t\}$ of a point configuration $\mathcal{A}$ is a collection of subsets $\sigma_i \subseteq [n], i = 1, \ldots, t$ such that

1. $\dim(\sigma_i) = d - 1$ for all $i = 1, \ldots, t$.
2. $\bigcup_{\sigma_i \in \Delta} \operatorname{conv}(\mathcal{A}_{\sigma_i}) = \operatorname{conv}(\mathcal{A})$.
3. For $i \neq j$, $\operatorname{conv}(\mathcal{A}_{\sigma_i}) \cap \operatorname{conv}(\mathcal{A}_{\sigma_j}) = \operatorname{conv}(\mathcal{A}_\tau)$ where $\tau = \sigma_i \cap \sigma_j$ is a common face of σ_i and σ_j.

Triangulations

Recall that a k-simplex is the convex hull of $k+1$ affinely independent points in $\mathbb{R}^d$. A **triangulation** of a point configuration $\mathcal{A}$ is a collection $\mathcal{T}$ of d-simplices, all of whose vertices are points in $\mathcal{A}$ such that

Triangulations

Recall that a k-simplex is the convex hull of $k+1$ affinely independent points in $\mathbb{R}^d$. A **triangulation** of a point configuration $\mathcal{A}$ is a collection $\mathcal{T}$ of d-simplices, all of whose vertices are points in $\mathcal{A}$ such that

1. The union of all these simplices equals $\mathrm{conv}(\mathcal{A})$.

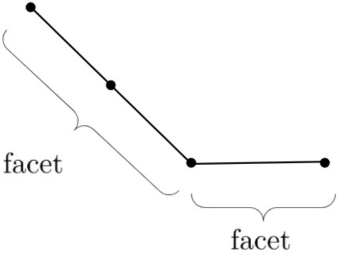
Triangulations

Recall that a k-simplex is the convex hull of $k+1$ affinely independent points in $\mathbb{R}^d$. A **triangulation** of a point configuration $\mathcal{A}$ is a collection $\mathcal{T}$ of d-simplices, all of whose vertices are points in $\mathcal{A}$ such that

1. The union of all these simplices equals conv$(\mathcal{A})$.
2. Any pair of these simplices intersects in a common face.

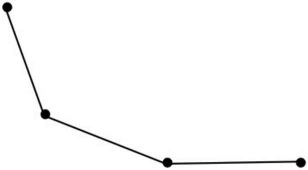
Triangulations

The sets $\{\sigma_i : i = 1, ..., t\}$ are the **facets** ($d - 1$ faces) of Δ and the indices that appear in the facets are called the **vertices** (0-faces).

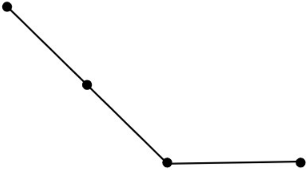


Triangulations

A triangulation in which every $i \in [n]$ is a vertex is called a **fine** triangulation.



fine



not fine

De Loera, et al. Ch. 7 Some Interesting Triangulations

The mother of all triangulations has the following point configuration.

$$\mathbf{M} := \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \end{array}$$

$$\mathbf{M} := \begin{pmatrix} 4 & 0 & 0 & 2 & 1 & 1 \\ 0 & 4 & 0 & 1 & 2 & 1 \\ 0 & 0 & 4 & 1 & 1 & 2 \end{pmatrix}$$

This point configuration and other similar configurations can be further studied in chapter 7 of De Loera, et al.

De Loera, et al. Ch. 7 Some Interesting Triangulations

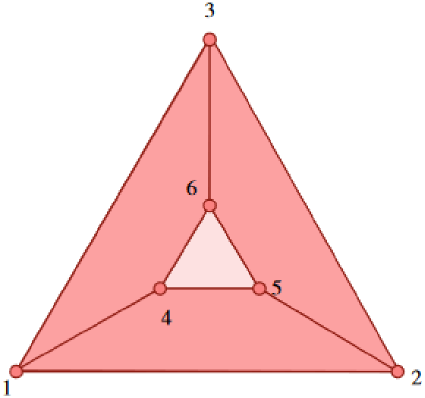


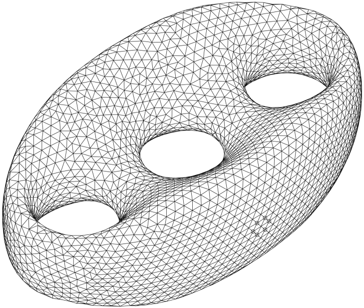
Figure 7.13: A subdivision of $\mathbf{M}$

Triangulations and Data Structures

Surfaces can be represented in code by using their triangulation. We can store vertices in a linear array and triangles in the nodes of a graph where edges connect neighboring triangles.

Triangulations and Data Structures

Surfaces can be represented in code by using their triangulation. We can store vertices in a linear array and triangles in the nodes of a graph where edges connect neighboring triangles. Some algorithms may benefit from such a structure. Also, we can use them to determine the topological type of a surface.



De Loera, et al. Ch. 6 Some Interesting Configurations

Definition

For $n > d \geq 0$, the d-dimensional **standard cyclic point configuration** with n points is the following point configuration:

$$\mathbf{C}(n,d) := \begin{array}{cccc} 1 & 2 & \dots & \mathrm{n} \\ \left(\begin{matrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{matrix}\right. & \begin{matrix} 2 \\ 1 \\ 2 \\ \vdots \\ 2^d \end{matrix} & \begin{matrix} \dots \\ \dots \\ \dots \\ \\ \dots \end{matrix} & \left.\begin{matrix} \mathrm{n} \\ 1 \\ \mathrm{n} \\ \vdots \\ \mathrm{n}^d \end{matrix}\right) \end{array}$$

The cyclic polytope is the convex hull of the points of $\mathbf{C}(n, d)$.

De Loera, et al. Ch. 6 Some Interesting Configurations

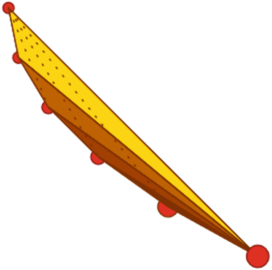


Figure 6.4: . . . and $\mathrm{conv}(\mathbf{C}(6,3))$ in dimension three.

De Loera, et al. Ch. 6 Some Interesting Configurations

Cyclic polytopes have many triangulations.

Theorem (6.1.22)

If d is fixed, the cyclic polytope $\mathrm{C}(n, d)$ *has at least* $\Omega(2^{\lfloor d/2 \rfloor})$ *triangulations.*

Cyclic polytopes are helpful for their "friendly" poset structure on the set of all triangulations of cyclic polytopes.

De Loera, et al. Ch. 6 Some Interesting Configurations

Below is a diagram of the triangulations of **C**$(6, 1)$ drawn as characteristic sections.

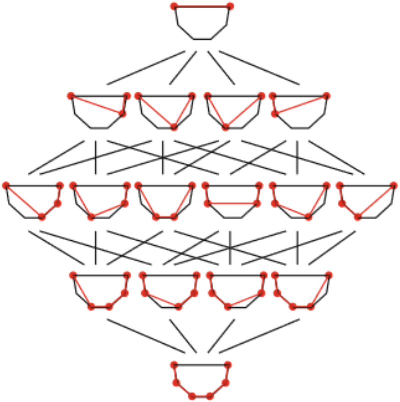


Figure 6.7: The height of a section defines a poset on all triangulations of $\mathbf{C}(6,1)$.

Regular Subdivisions

Given a graded point configuration in $\mathbb{R}^d$, we can “lift” it into $\mathbb{R}^{d+1}$ using a **weight vector**.

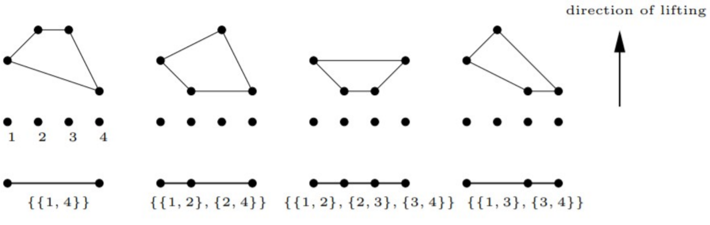


Figure 2. Regular triangulations.

Regular Subdivisions

Let $P^\omega \subset \mathbb{R}^{d+1}$ be the convex hull of this lifted point configuration. Then, projecting the "lower" faces of P^ω back onto $\mathcal{A}$ induces a subdivision of $\mathcal{A}$, which we denote Δ_ω and call a **regular subdivision**.

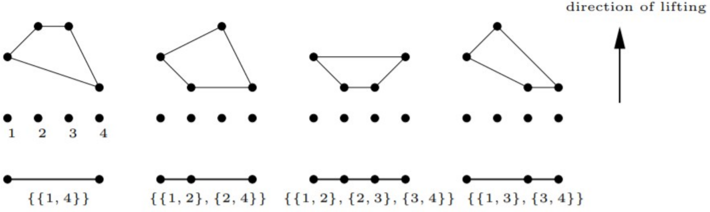


Figure 2. Regular triangulations.

Cones

A **cone** $K \subseteq \mathbb{R}^d$ is any subset of $\mathbb{R}^d$ such that

1. For $x, y \in K$, $x + y \in K$.
2. For $x \in K$ and $\lambda \geq 0$, $\lambda x \in K$.

Cones

A **polyhedral cone** $K \subseteq \mathbb{R}^d$ is a polyhedron of the form

$$K = \{\mathbf{x} \in \mathbb{R}^d : M\mathbf{x} \geq \mathbf{0}\}$$

where M is a real matrix.

Cones

A polyhedral cone can also be thought of as the set of all non-negative combinations of columns of some real matrix N

$$K = \{N\mathbf{y} : \ \geq \mathbf{0}\}$$

Theorem

Every finitely generated constrained cone is a finitely generated cone, and vice-versa.

Cone Complexes

Definition

A **cone complex**, or more typically a **polyhedral fan**, is a polyhedral complex in which all the polyhedra are cones.

Normal Cones

Let $P \subset \mathbb{R}^d$ be a polyhedron and F be a face of P. The **outer normal cone** of P at F is defined as

$$\mathcal{N}_P(F) = \{\mathbf{c} \in \mathbb{R}^d : F = \mathrm{face}_{\mathbf{c}}(P)\}.$$

The collection of outer normal cones of P is called the **outer normal fan** of P.

Normal Cones

Let $P \subset \mathbb{R}^d$ be a polyhedron and F be a face of P. The **outer normal cone** of P at F is defined as

$$\mathcal{N}_P(F) = \{\mathbf{c} \in \mathbb{R}^d : F = \mathrm{face}_{\mathbf{c}}(P)\}.$$

The collection of outer normal cones of P is called the **outer normal fan** of P.

The **inner normal cone** of P at F is defined as

$$\mathcal{N}_P(F) = \{\mathbf{c} \in \mathbb{R}^d : F = \mathrm{face}_{-\mathbf{c}}(P)\}.$$

Likewise, the collection of inner normal cones of P is called the **inner normal fan** of P.

Normal Cones

Let $P \subset \mathbb{R}^d$ be a polyhedron and F be a face of P. The **outer normal cone** of P at F is defined as

$$\mathcal{N}_P(F) = \{\mathbf{c} \in \mathbb{R}^d : F = \text{face}_{\mathbf{c}}(P)\}.$$

The collection of outer normal cones of P is called the **outer normal fan** of P.

The **inner normal cone** of P at F is defined as

$$\mathcal{N}_P(F) = \{\mathbf{c} \in \mathbb{R}^d : F = \text{face}_{-\mathbf{c}}(P)\}.$$

Likewise, the collection of inner normal cones of P is called the **inner normal fan** of P.

Thomas notes that the notation is the same in the book as they will not be referenecd simultaneously in the book.

Complete and Incomplete Fans

Definition

If the support of the fan is the entire space it lives in, we call it a **complete fan**. Else, we say it is an **incomplete fan**.

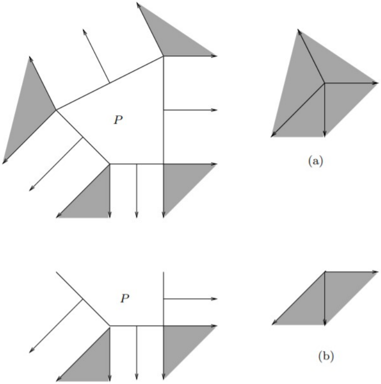


Figure 4. A complete and incomplete fan.

Exercise 7.8

Exercise 7.8. Prove that both the inner and outer normal fans of a polyhedron are cone complexes. (You will need to show that the intersection of two cones in a fan is a common face of each and that every face of every cone in a fan is again a cone in the fan.)

Exercise 7.22

Exercise 7.22. Let $\mathcal{A}$ be a point configuration with 6 points in $\mathbb{R}^3$ given in Example. 7.12. Carefully describe its refinement poset. Indicate which subdivisions Δ are regular by giving a weight vector ω that induces each regular subdivision.

Gale Transform

Let $B_1, \ldots, B_{n-d} \in \mathbb{R}^n$ be a basis for the vector space $\ker_{\mathbb{R}}(A)$. If we organize these vectors as the columns of an $n \times (n-d)$ matrix

$$B := (B_1 \quad B_2 \quad \cdots \quad B_{n-d}),$$

we see that $AB = \mathbf{0}$.

Gale Transform

Definition

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\} \subset \mathbb{R}^{n-d}$ be the n ordered rows of B. Then $\mathcal{B}$ is called **Gale transform** of $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. The associated **Gale diagram** of $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is the *vector configuration* $\mathcal{B}$ drawn in $\mathbb{R}^{n-d}$.

Theorem of Chapter 7

Theorem

Let $\triangle = \{\sigma_1, \ldots, \sigma_t\}$ *be a subdivision of* $\mathcal{A}$ *and* $\mathcal{B}$ *be a Gale transform of* $\mathcal{A}$*. Then* $\triangle$ *is regular if and only if*

$$\bigcap_{i=1}^{t} \operatorname{relint}(\operatorname{cone}(\mathcal{B}_{\overline{\sigma}_i})) \neq \emptyset.$$