MATH 4320 Project: Fluid Flow

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1 Motivation

The tools of conformal mapping enables us to model different scenarios of fluid flow.

2 Two-Dimensional Fluid Flow

In this summary we will go over two important concepts (velocity potential and the stream function) in simplified three dimensional fluid dynamics by considering the flow in two dimensions and assuming that the motion in any plane parallel to the xy-plane is the same without any sources or sinks. To do this, we must assume three important factors. In the scenarios that we will cover, the fluids will be **irrotational** where angular speed of the fluid is zero, **incompressible** where density is the same, and **free from viscosity** which is a measure of the fluid's resistance as a function of the relative velocity of its particles, at every point (x, y) in the domain considered. These approximations are sufficient in several applications of fluids.

Begin with defining the **velocity** of a fluid to be:

$$V(x, y) = p(x, y) + iq(x, y)$$

where (x, y) is a point on the plane. Then, **circulation** is defined as the line integral over some contour C with respect to arc length σ of the tangential component of velocity along C:

$$\int_C V_T(x, y) d\sigma.$$

Further, we can enumerate the **rotation** of the fluid, which can be understood as the angular speed of the fluid at center (x, y). To do this, let

$$\int_C V_T(x, y) d\sigma = \int_C p(x, y)dx + q(x, y)dy.$$

Then, apply Green's theorem to achieve rectangular integration bounds:

$$\int_C p(x, y)dx + q(x, y)dy = \int\int_R [q_x(x, y) - p_y(x, y)]dA.$$

Now, we are able to represent circulation as an integral over mean speeds of points along C. If C is a circle, we can divide the integral by the circumference, $2\pi r$, where r is the radius. Then, since $speed = radius \cdot angularspeed$, we divide the integral further by r. Hence, we have

$$\frac{1}{\pi r^2} \int\int_R \frac{1}{2}[q_x(x, y) - p_y(x, y)]dA.$$

which is the expression for the mean value of the function $\omega(x, y) = \frac{1}{2}[q_x(x, y) - p_y(x, y)]$, which is the rotation of the fluid at center (x, y).

Another important function in studying fluids is the velocity potential. Let D be an irrotational domain and C be any simply connected contour in D. Since the fluid is irrotational, $0 = \frac{1}{2}[q_x(x,y) - p_y(x,y)] \implies q_x = p_y$. Thus, $\int_C p(s,t)ds + q(s,t)dt$ for a contour C joining two points (x_0, y_0) and (x, y) is actually independent of path. Then, denote

$$\phi(x,y) = \int_{(x_0,y_0)}^{(x,y)} p(s,t)ds + q(s,t)dt$$

to be the **velocity potential**. Note that the gradient of the velocity potential is equal to the velocity of the flow and since the fluid is incompresssible and there are no sources or sinks, it must also satisfy Laplace's equation

$$\phi_{xx}(x,y) + \phi_{yy}(x,y) = 0.$$

The level curves $\phi(x,y) = c$ for a constant c are called **equipotentials**.

For a simply connected irrotational domain, $V = \phi_x(x,y) + i\phi_y(x,y) = \phi(x,y)$ as stated above. Denote $\psi(x,y)$ to be the harmonic conjugate of $\phi(x,y)$. ψ is called the **stream function** and the curves of the ψ where its output is constant are called **streamlines** of the flow.

The **complex potential** of the flow is the function

$$F(z) = \phi(x,y) + i\psi(x,y).$$

It is useful to use the complex potential of flows to extract velocity and rate of flow at points in the domain by taking partial derivatives.

For instance by the Cauchy-Riemann equations,

$$F'(z) = \phi_x(x,y) - i\phi_y(x,y) \implies V = \overline{F'(z)}.$$

Interestingly, since ϕ is harmonic in the simply connected domain D, ψ can be written as

$$\psi(x,y) = \int_{(x_0,y_0)}^{(x,y)} p(s,t)ds - q(s,t)dt = \int_C V_N(x,y)d\sigma$$

where $V_N(x,y)$ is the normal component of velocity at point (x,y). Then, we can understand $\psi(x,y)$ to be the rate of flow across the xy-plane on a contour C.

Using the velocity potential and the stream function in a combined manner through the complex potential above, we can now model flows around a corner and flows around a cylinder. The key to these scenarios is applying the correct transformation. Transforming the complex potential into these forms allows for the correct calculation for velocity and rate of flow.

2.1 Example flow with corner

In the case of a downward flow meeting a corner in the first quadrant, the transformation would be $w = z^2 = x^2 - y^2 + i2xy$. Then, $F(x,y) = A[w(x,y)]^2$ for some constant A. Then, it becomes clear that $\psi(x,y) = 2Axy$ and $|V| = \overline{F'(x,y)} = 2A\sqrt{x^2+y^2}$.

2.2 Example flow with cylinder

In the case of a cylinder centered at the origin, the transformation would be $w = z + \frac{1}{z}$. Then, $F(z) = A(z + \frac{1}{z})$. Thus, $V = A(1 - \frac{1}{z^2})$ and $\psi = A(r - \frac{1}{r})\sin\theta$.

2.3 Example flow in channel through a slit

In the case of flow through a slit in a channel, let our transformation be $w = \text{Log}z$. Then, the width of the channel in the uv-plane will be π and the slit is located at $u = 0$. Let $Q = \psi(u_1, 0)$ for $u_1 < 0$ be the rate of flow into the channel through the slit.

2.3.1 Rate of flow

Then,

$$\psi = \frac{Q}{\pi}\Big[\text{Arg}(z-1) - \frac{1}{2}\text{Arg}z\Big].$$

2.3.2 Velocity

$$V = \frac{Q}{2\pi}\coth\frac{\overline{w}}{2}.$$

2.3.3 Stagnation Points

Stagnation points are defined to be points in the domain where the velocity is zero. There is a stagnation point at $w = \pi i$.

2.3.4 Streamlines

The streamlines of the model satisfy

$$\psi(u,v) = c^2 = \frac{Q}{\pi}\text{Arg}\Big(\sinh\frac{w}{2}\Big).$$

2.4 Example flow in channel with an offset

This scenario utilizes the Schwarz-Christoffel transformation since the offset will bring a change in breadth that can be modeled using a transformation along a polygon. Consider the figure below from Sec. 132.

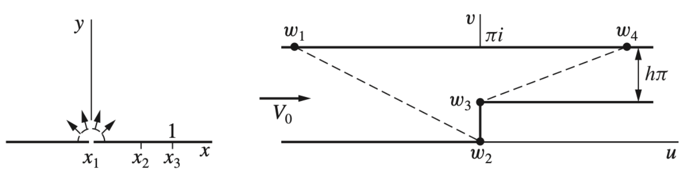

FIGURE 188

To model this scenario correctly, we take w_1 and w_4 to be infinitely far left and far right respectively. Thus, we can write $x_1 = 0, x_3 = 1, x_4 = \infty$ and $0 < x_2 < 1$ to be determined, the mapping function would be

$$\frac{dw}{dz} = Az^{-1}(z-x_2)^{-1/2}(z-1)^{1/2}$$

for some constants A and x_2.

2.4.1 Complex Potential

Similarly to the previous scenario,

$$F = V_0\text{Log}z = V_0\ln r + iV_0\theta.$$

2.4.2 Velocity

$$\overline{V(w)} = \frac{V_0}{A}\Big(\frac{z-x_2}{z-1}\Big)^{1/2}.$$

3 Exercises

126.1. State why the components of velocity can be obtained from teh stream function by means of the equations $p(x, y) = \psi_y(x, y); q(x, y) = -\psi_x(x, y)$.

126.2. At an interior point of a region of flow and under the conditions that we have assumed, the fluid pressure cannot be less than the pressure at all other points in a neighborhood of that point. Justify this statement with the aid of statements in Secs. 124, 125, and 59.

126.3. For the flow around a corner described in Example 1, Sec. 126, at what point of the region $x \geq 0, y \geq 0$ is the fluid pressure greatest?

126.4. Show that the speed of the fluid at points on the cylindrical surface in Exapmle 2, Sec. 126, is $2A|\sin\theta|$ and also that the fluid pressure on the cylinder is greatest at the poitns $z = \pm 1$ and least at the points $z = \pm i$.

126.5. Write the complex potential for the flow around a cylinder $r = r_0$ when the velocity V at a point z approaches a real constant A as the point recedes from the cylinder.

126.6. Obtain the stream function $\psi = Ar^4 \sin 4\theta$ for a flow in the angular region $r \geq 0, 0 \leq \theta \leq \frac{\pi}{4}$ that is shown in Fig. 175. Sketch a few of the streamlines in the interior of that region.

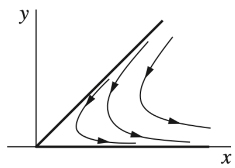

FIGURE 175

126.7. Obtain the complex potential $F = A \sin z$ for a flow inside the semi-infinite region $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}, y \geq 0$ that is shown in Fig. 176. Write the equations of the streamlines.

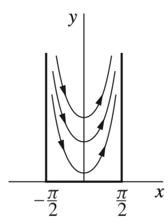

FIGURE 176

126.8. Show that if the velocity potential is $\phi = A \ln r (A > 0)$ for flow in the region $r \geq r_0$, then the streamlines are the half lines $\theta = c(r \geq r_0)$ and the rate of flow outward through each complete circle about the origin is $2\pi A$, corresponding to a source of that strength at the origin.

126.9. Obtain the complex potential $F = A(z^2 + \frac{1}{z^2})$ for a flow in the region $r \geq 1, 0 \leq \theta \leq \pi/2$. Write expressions for V and ψ. Note how the speed $|V|$ varies along the boundary of the region, and verify that $\psi(x, y) = 0$ on the boundary.

132.2. Explain why the solution of the problem of flow in a channel with a semi-infinite rectangular obstruction (Fig. 191) is included in the solution of the problem treated in Sec. 121.

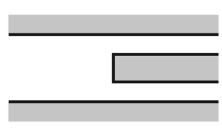

FIGURE 191

132.5. Let $F(w)$ denote the complex potential function for the flow fo a fluid over a step in the bed of a deep stream represented by the shaded region of the w plane if Fig. 29, Appendix 2, where the fluid velocity

V approaches a real constant V_0 as $|w|$ tends to infinity in that region. The transformation that maps the upper half of the z plane onto the region is noted in Exercise 3. Use the chain rule

$$\frac{dF}{dw} = \frac{dF}{dz}\frac{dz}{dw}$$

to show that

$$\overline{V(w)} = V_0(z-1)^{1/2}(z+1)^{-1/2};$$

and, in terms of the points $z = x$ whose images are the points along the bed of the stream, show that

$$|V| = |V_0|\sqrt{\left|\frac{x-1}{x+1}\right|}.$$

Note that the speed increases from $|V_0|$ along $A'B'$ until $|V| = \infty$ at B', then diminishes to zero at C', and increases toward $|V_0|$ from C' to D'; note, too, that the speed is $|V_0|$ at the point

$$w = i(\frac{1}{2} + \frac{1}{\pi})h,$$

between B' and C'.

3.1 Solutions

126.1. In a simply connected domain, we note that the complex potential $F(z) = \phi(x,y) + i\psi(x,y)$ is analytic throughout. Thus, $F'(z) = \phi_x(x,y) + i\psi_x(x,y) = \psi_y(x,y) - \phi_x(x,y)$ by Cauchy-Riemann equations and $V = p(x,y) + iq(x,y)$. Thus, $p(x,y) = \psi_y(x,y)$ and $q(x,y) = -\psi_x(x,y)$.
126.2. By way of contradiction, suppose that by Bernoulli's theorem $P(z) = \rho c - \frac{\rho}{2}|V(z)|^2 > P(z_0)$ for some constant c and constant density ρ at some neighborhood of z_0. Let $g(z) = 1/P(z)$. Note, $P(z)$ is analytic if $V(z)$ is analytic in the domain. Then, our problem statement becomes "$g(z)$ cannot be greater at one interior point than any other points in the neighborhood of that point." By the maximum modulus principle, however, we know that if pressure is not constant throughout the domain and since $g(z)$ is analytic, then $|g(z)|$ has no maximum within the domain, and so there does not exist any point such that the pressure is less than the pressure at all other points in a neighborhood of that point.
126.3. Bernourlli's principle states that $\frac{P}{\rho} + \frac{1}{2}|V|^2 = c$ for some constant c. Since we are considering incompressible fluids, ρ is also constant. Thus, pressure is greatest when velocity is least and in our problem that is at the point $z = 0$.
126.4. From our notes above, we know that in the case of a cylinder, $V(z) = A(1 - 1/\overline{z}^2)$. Thus, we wish to show that $|1 - 1/\overline{z}^2| = 2|\sin\theta|$. Let $\overline{z} = re^{-i\theta}$ and since we are considering the cylinder in example 2, $r = 1$. Then, $|1 - 1/\overline{z}^2| = |1 - e^{2i\theta}| = |1 - \cos 2\theta - i\sin 2\theta|$. If we take the modulus of the RHS, we have $\sqrt{(1-\cos 2\theta)^2 + (\sin 2\theta)^2} = \sqrt{4\frac{1-\cos 2\theta}{2}} = 2\sqrt{\frac{1-\cos 2\theta}{2}}$. This is exactly the expression for $|2\sin\theta|$. Thus, $|V(z)| = 2A|\sin\theta|$.
126.5. We will map an r_0 cylinder to our typical $r = 1$ cylinder. Let $Z = z/r_0$ and then $w = Z + 1/Z = z/r_0 + r_0/z$. Then, we initially have $F(z) = c(z/r_0 + r_0/z)$ for some constant c. Since $V = \overline{F'(z)}$, we have $V = c/r_0 - r_0/z^2$. Taking the limit as r goes to ∞, we have

$$\lim_{r\to\infty} V = A = \lim r \to \infty c/r_0 - r_0/z^2 = c/r_0.$$

Thus, writing $F(z)$ in terms of A,

$$F(z) = A(z + r_0^2/z).$$

126.6. Naturally, we consider the transformation $w = z^4$. Then, $w = r^4e^{i4\theta} = r^4\cos 4\theta + ir^4\sin 4\theta$. Thus, $F(z) = Az = \phi(z) + i\psi(z) = Ar^4\cos 4\theta + iAr^4\sin 4\theta$ and so the stream function $\psi = Ar^4\sin 4\theta$. The streamlines will look exactly like those in the figure below.

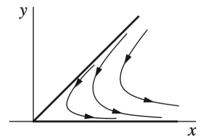

FIGURE 175

126.7. Let our transformation be $w = \ sinz$. Then, $F = Aw = A \sin z$. Further, $Aw = A(u+iv) = A\sin z = A(\frac{e^{iz}-e^{-iz}}{2i})$. Letting $z = x + iy$,

$$A(u+iv) = \frac{A}{2i} \cdot (e^{ix-y} - e^{-ix+y}) = \frac{A}{2i} \cdot (e^{-y}(\cos ix + i \sin ix) - e^{y}(\cos -ix + i \sin -ix)).$$

Substituting for hyperbolic trig functions, we have $Au + iAv = A \sin x \cosh y + iA \cos x \sinh y$. Thus, the streamlines of this flow satisfy $c = A \cos x \sinh y$ for some constant c.
126.8. Letting our complex potential be $F(z) = A\text{Log}z$, we have $F(z) = A(\ln r + i\theta)$. Thus, $\phi = A \ln r$ and $\psi = A\theta$. Our streamlines must then satisfy $c = \theta$ for some constant c. Since the stream function is $A\theta$, we let $\theta = 2\pi$ to be the complete circle about the origin, and so the rate of flow is $2\pi A$.
126.9. Let our complex potential be $F(z) = A(z^2 + 1/z^2)$. Then,

$$\begin{aligned} V &= \overline{F'(v)} \\ &= \overline{A2z - 2A/z^3} \\ &= A2\overline{z} - 2A/\overline{z}^3 \end{aligned}$$

Finding ψ, we let $z = re^{i\theta}$. Then, $F(z) = Ar^2e^{i2\theta} + Ar^{-2}e^{-2i\theta} = Ar^2 \cos 2\theta + iAr^2 \sin 2\theta + Ar^{-2} \cos -2\theta + iAr^{-2} \sin -2\theta$. Thus, $\Im(F(z)) = \psi = A(r^2 - r^{-2}) \sin 2\theta$. On the boundary, $r = 1$, and so $\psi(1e^{i\theta}) = 0$.

Further on the boundary, $r = 1$ and theta varies from 0 to $\pi/2$. Thus, given our transformation, $\overline{F'(z)}$ must vary, and so the speed varies along the boundary.
132.2. By the Schwarz-Christoffel transformation, our limiting positions, ω_2, ω_3, and ω_4 are those boundary points considered in the problem of Sec. 121. Thus, the two are related by ways of conformal mapping.
132.5. Let our complex potential be $F(z) = cz$. Given that our transformation is $w = \frac{h}{\pi}[(z^2-1)^{1/2} + \cosh^{-1} z]$, then

$$\frac{dw}{dz} = \frac{h}{\pi} z(z^2-1)^{-1/2} + \frac{h}{\pi}(z^2-1)^{-1/2} = \frac{h}{\pi} \cdot \sqrt{\frac{1}{(z-1)(z+1)}} \cdot \sqrt{(z+1)^2} = \frac{h}{\pi}\sqrt{\frac{z+1}{z-1}}.$$

Then, $V(w) = \overline{\frac{dF}{dw}}$. Thus, $\overline{V(w)} = \frac{dF}{dz}\frac{dz}{dw} = c \cdot (\frac{dw}{dz})^{-1} = \frac{c\pi}{h}\sqrt{\frac{z-1}{z+1}}$. Thus, $\lim_{|w|\to\infty} V(w) = \lim_{z\to\infty} V(z) = \frac{c\pi}{h}\sqrt{\frac{1-1/z}{1+1/z}} = \frac{c\pi}{h}$. Then, $V_0 = \frac{c\pi}{h}$ and so $\overline{V(w)} = V_0\sqrt{\frac{z-1}{z+1}}$ as desired. Further, for the points $z = x$ whose images are the points along the bed of the stream,

$$|V| = V_0\sqrt{\frac{x-1}{x+1}}.$$

Plugging in $w_0 = i(\frac{1}{2} + \frac{1}{\pi})h$ into $V(w)$ from above, we have $V(w_0) = V_0$.

4 Applications

The obvious application of studying two dimensional fluid flows are that we can scale them to three dimensional fluid dynamics. It seems to me that we will likely be successful in modeling fluid dynamics in three dimensions when we examine a large enough scale such that varieties in fluid density and viscosity become negligible. For example, applying our formulas to capillary action would likely result in greater error than if we were to examine flows in rivers. Further, we have yet to consider more turbulent flows, where there might be numerous sources and sinks for example, that would create more unpredictability. Considering these factors would help draw greater breadth in our understanding of fluid flows.

When considering turbulence, one of the famous extensions of the problem with fluid flows is the Navier-Stokes existence and smoothness problem. The Navier-Stokes equations are a series of PDEs that describe the motion of a fluid in $\mathbb{R}^n$ and are to be solved for some unknown velocity and pressure vector for some point in $\mathbb{R}^n$ and time t. While there are more variables to consider than our textbook goes over, the initial assumptions stay relatively the same. In the context of this millenium problem, we assume that the fluid is incompressible, Newtonian, and isothermal.

Further, we can consider the effects of heat on fluids. It is commonly known that heat affects the density of fluids, and so one potential topic to study is modelling convection in fluids. For example, we can study the ideal gas law and how temperature is related to pressure, and from above, we know that pressure is related to fluid velocity.