The Conway Knot is Not Slice

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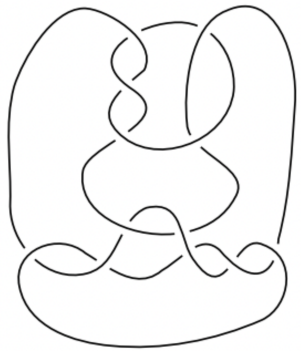
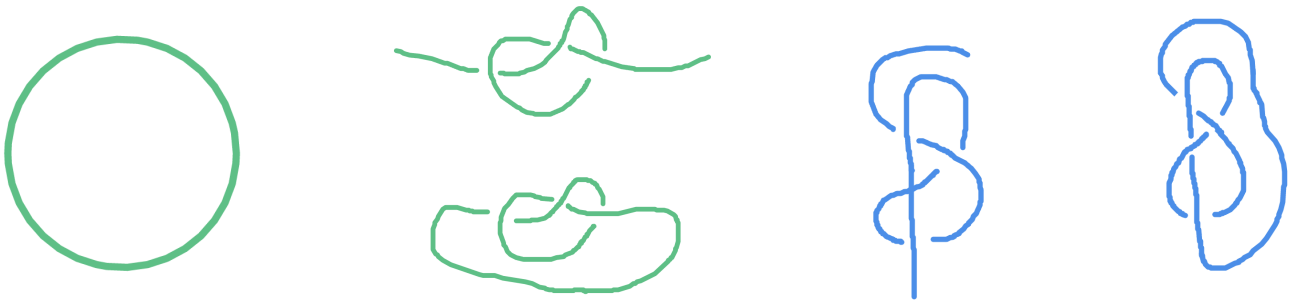


Knots

Definition

A knot is an embedding of a circle (S^1) into a three-dimensional sphere S^3



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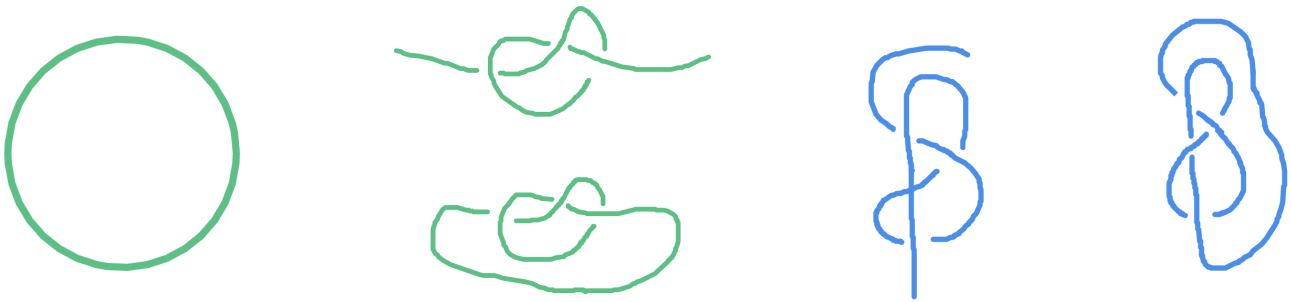
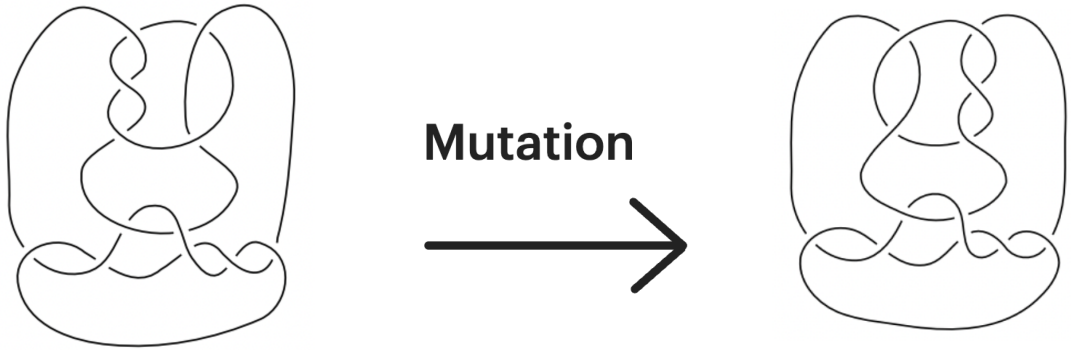


Definition

A knot is *slice* if it bounds a smoothly embedded disk in B^4

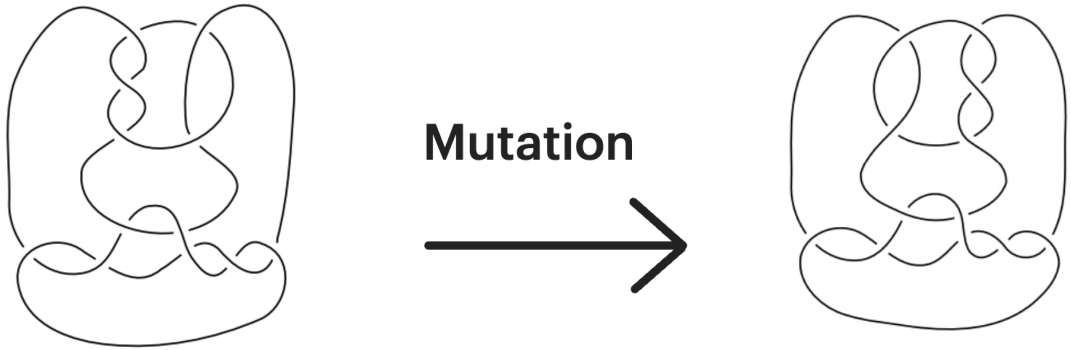
- Rasmussen's s-invariant, $s(K) \neq 0$ if K is not slice.

Conway Knot



- The Conway knot is famous for being elusive. Four years ago, all slice knots with crossing number < 13 were known except for the Conway knot having 11 crossings.
- By mutation we get the Kinoshita-Terasaka knot.

Conway Knot



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Theorem (Piccirillo)

The Conway knot is not slice

Proof Strategy

1. Contruct a knot K that shares a zero trace as the Conway knot. Very involved.

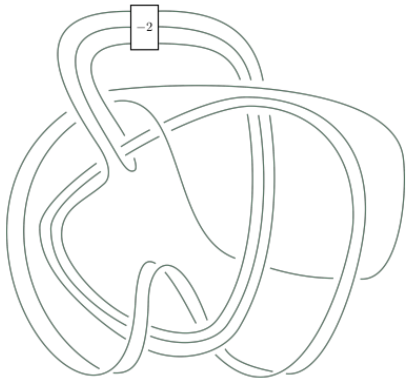

FIGURE 2. The knot K' shares a trace with the Conway knot (1)

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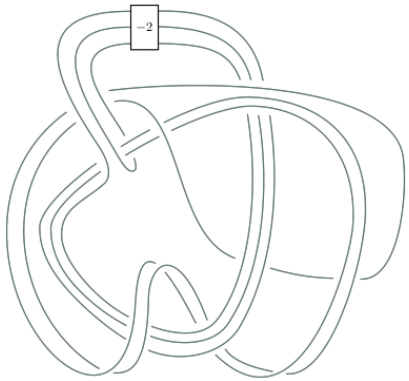

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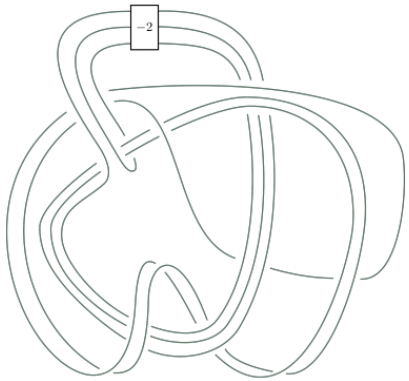

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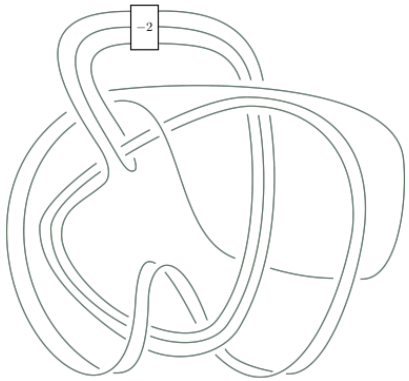

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Trace Embedding Lemma

Lemma

K is slice if and only if $X_0(K)$ smoothly embeds in S^4

- Proof will use handle attachments and cones of knots

Definition

A *knot trace* $X_0(K)$ is a 4-manifold obtained by attaching a 0-framed 2-handle to B^4 with attaching sphere K.

What's Next?

- Determine which knots are slice with 13 and 14 crossings
- Use tools to study smooth 4-dimensional Poincaré conjecture

References

[1] Lisa Piccirillo The Conway Knot is Not Slice. *Annals of Mathematics Vol. 191 (2020), 581-591.*